A Problem of Probability in a Game of Darts

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Introduction Certain variations of the dart game known as “Round the World” or “Round the Clock” present the interesting possibility of a player winning the game without the other player throwing a single dart. A natural question to ask is how likely this is to happen. The basic game is played by two players who alternate turns consisting of three darts. The goal is to hit each of the twenty sectors of the dartboard, which are numbered 1 through 20, in order. The first player to hit all twenty is the winner. A variation on this is to allow a player to take an extra turn (i.e. *not* to alternate turns) if a sufficient number of target sectors are hit during the player’s turn. If the first player is accurate enough, he may win without the second player throwing any darts, by repeatedly being allowed extra turns. Let’s call this a *perfect* win. If we assume that each throw of a dart has the same probability p of hitting its target sector, what is the probability that the first player will have a perfect win?

First we note some useful probabilities. Let $q = 1 - p$ (so that q is the probability of missing a targeted sector). Then the probability of hitting three targeted sectors in a single turn is p^3, hitting (exactly) two sectors is $3p^2q$, hitting (exactly) one sector is $3pq^2$, and hitting no sectors is q^3.

There are three variations to consider. The player may be allowed an extra turn if at least one, two, or three target sectors are hit during a single turn.

First Variation The variation in which the player needs to hit all three target sectors in order to take another turn is the simplest to deal with. To win without the second player ever getting a turn, the first player must hit all of the sectors 1 through 18 without missing and then, on the final turn, hit sectors 19 and 20. The probability of hitting sectors 19 and 20 in a single turn is

$$p^2 + 2p^2q = p^2\left(1 + 2q\right).$$

Thus, the probability of the first player having a perfect win is (we’ll call it $f_3(p)$)

$$f_3(p) = \left(p^3\right)^6\left(p^2\left(1 + 2q\right)\right) = p^{20}\left(1 + 2q\right) = p^{20}\left(3 - 2p\right).$$

This is, as we would expect, an increasing function of p, and has positive second derivative for $0 < p \leq 1$. It is less than p for $0 < p < 1$.

Second Variation The other variations are a bit more complicated. In the second variation, the first player is allowed to miss one throw on each turn and still take an extra turn. That is, on each turn, the player gets an extra turn exactly when two or three darts hit their targets. This allows a great number of ways for the first player to achieve a perfect win. As a result, the calculation involved looks frightfully tedious. However, we are saved from this tedium (most of it, anyway) by two things: treating the game as a *Markov process* (see, e.g., [**2** chapter 3]), and the use of a computer algebra system. To treat the game as a Markov process, we consider the player to be in one of a finite number of states, and we determine the probability of transitioning from one state to another on each turn. Here we will consider 22 states corresponding to the twenty sectors, plus a "lost" state L and a "won" state W. The first twenty states correspond to the first player's current target. State L is the state at which the player has lost his turn, and the second player will get to throw. The "won" state is the state that the player reaches after hitting sector 20.

An example may help make this clear. The first player starts in state 1, since sector 1 is the first target. Suppose that on the first turn, sector 1 is struck on the first throw, sector 2 is missed on the second throw, and sector 2 is struck on the third throw. Then the player gets another turn, and since 3 is the next target, we consider the player to be in state 3. That is, the player has *transitioned* from state 1 to state 3 in the first turn. If the player misses two or three times on the next turn, then the player moves to the "lost" state L, since the player will not get another turn before the other player throws. Thus the transition would be from state 3 to state L. This process continues for 10 turns, since the player cannot take more than 10 turns to get to state W. If a player ends up in state L or state W before the 10th turn, then the player stays in that state. What we seek is the probability that the player will be in state W after 10 turns.

To calculate this, we will create a *transition matrix*, A, with entries $a_{i,j}$, where $a_{i,j}$ is the probability of transitioning from state i to state j in a single turn. Let state 21 represent the W state, and state 22 represent the L state. For $i \leq 18$,

$$a_{i,j} = \begin{cases} 0 & \text{if } j \leq i+1 \ ; \\ 3p^2q & \text{if } j = i+2 \ ; \\ p^3 & \text{if } j = i+3 \ ; \\ 1-3p^2q-p^3 & \text{if } j = 22; \\ 0 & \text{otherwise.} \end{cases}$$

Further,

$$a_{19,j} = \begin{cases} 0 & \text{if } j \leq 20; \\ p^2+2p^2q & \text{if } j = 21; \\ 1-p^2-2p^2q & \text{if } j = 22; \end{cases} \qquad a_{20,j} = \begin{cases} 0 & \text{if } j \leq 20; \\ p+pq+pq^2 & \text{if } j = 21; \\ 1-p-pq-pq^2 & \text{if } j = 22; \end{cases}$$

$$a_{21,j} = \begin{cases} 0 & \text{if } j \neq 21; \\ 1 & \text{if } j = 21; \end{cases} \qquad \text{and} \qquad a_{22,j} = \begin{cases} 0 & \text{if } j \neq 22; \\ 1 & \text{if } j = 22. \end{cases}$$

Hence, matrix A looks like this:

$$A = \begin{pmatrix}
0 & 0 & 3p^2q & p^3 & 0 & \dots & \dots & \dots & \dots & 0 & 1-3p^2q-p^3 \\
0 & 0 & 0 & 3p^2q & p^3 & 0 & \dots & \dots & \dots & 0 & 1-3p^2q-p^3 \\
0 & 0 & 0 & 0 & 3p^2q & p^3 & 0 & \dots & \dots & 0 & 1-3p^2q-p^3 \\
0 & 0 & 0 & 0 & 0 & 3p^2q & p^3 & 0 & \dots & 0 & 1-3p^2q-p^3 \\
\vdots & & & & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots \\
0 & \dots & \dots & \dots & \dots & \dots & 0 & 3p^2q & p^3 & 0 & 1-3p^2q-p^3 \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 3p^2q & p^3 & 1-3p^2q-p^3 \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & p^2(1+2q) & 1-p^2(1+2q) \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & p(1+q+q^2) & 1-p(1+q+q^2) \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 0 \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1
\end{pmatrix}.$$

Now, the probability of transitioning from state 1 to the winning state in 2 moves is

$$a_{1,1}a_{1,21} + a_{1,2}a_{2,21} + \cdots + a_{1,21}a_{21,21}.$$

This can be calculated from A: the probability is $(A^2)_{1,21}$ (which is, of course, identically zero). Similarly, the probability we want is

$$\sum a_{1,j_1}a_{j_1,j_2}\cdots a_{j_8,j_9}a_{j_9,21}$$

where the sum is over all $9-$tuples of integers $(j_1, ..., j_9)$, $1 \le j_i \le 22$; this is $(A^{10})_{1,21}$. The work of taking the 10-th power of a 22 by 22 matrix is greatly reduced if we use a computer algebra system - all we have to do is carefully enter the matrix and the proper exponentiation command. Expressed as a function of p, the probability of a perfect win is

$$f_2(p) = (A^{10})_{1,21} = p^{20}(362010 - 2964242p + 11106342p^2 - 25084782p^3 + 37821546p^4$$

$$-39755772p^5 + 29471526p^6 - 15188472p^7 + 5196312p^8 - 1062882p^9 + 98415p^{10}).$$

This function is strictly increasing from 0 to 1, has an inflection point near 0.841351 and is equal to p at approximately 0.950087. The 10-th degree polynomial above is irreducible by Eisenstein's criterion (see, e.g. Garling [**1**, p.51]): all of the coefficients except the final one are even, and 4 does not divide 362010.

Third Variation The last case to consider is when the first player need only hit one sector on a turn in order to take another turn. This is similar to the previous case, except that the transition matrix is less sparse, since the transition from state i to state $i + 1$ is now possible.

Also, the player may take up to 20 turns to reach 20, and so our matrix will be raised to the 20th power (instead of 10). This results in a polynomial, $f_1(p)$, of degree 60, with a largest coefficient of 576036001898562109302. This function is strictly increasing for $0 < p < 1$, equals p at approximately $p = 0.6544018$, and has an inflection point near $p = 0.58911198$.

Comparison and Comment The following table gives the values of $f_1(p)$, $f_2(p)$, and $f_3(p)$ for various values of p.

p	$f_1(p)$	$f_2(p)$	$f_3(p)$
0.001	3.97×10^{-49}	3.59×10^{-55}	3.00×10^{-60}
0.01	3.08×10^{-29}	3.33×10^{-35}	2.98×10^{-40}
0.05	9.48×10^{-16}	2.28×10^{-21}	2.77×10^{-26}
0.10	2.38×10^{-10}	1.55×10^{-15}	2.80×10^{-20}
0.15	1.87×10^{-7}	3.29×10^{-12}	8.98×10^{-17}
0.20	1.37×10^{-5}	6.53×10^{-10}	2.73×10^{-14}
0.25	2.76×10^{-4}	3.49×10^{-8}	2.27×10^{-12}
0.30	0.00245	8.11×10^{-7}	8.37×10^{-11}
0.35	0.01237	1.05×10^{-5}	1.75×10^{-9}
0.40	0.04173	8.80×10^{-5}	2.42×10^{-8}
0.45	0.10405	5.26×10^{-4}	2.43×10^{-7}
0.50	0.20607	0.00239	1.91×10^{-6}
0.55	0.34177	0.00865	1.22×10^{-5}
0.60	0.49404	0.02573	6.58×10^{-5}
0.65	0.64217	0.06453	3.08×10^{-4}
0.70	0.76943	0.13887	0.00128
0.75	0.86688	0.26006	0.00476
0.80	0.93330	0.42829	0.01614
0.85	0.97284	0.62519	0.05039
0.90	0.99231	0.81376	0.14589
0.95	0.99909	0.94992	0.39433
0.99	0.999993	0.99792	0.83427

As we'd expect, $f_1(p) > f_2(p) > f_3(p)$ for $0 < p < 1$. The table shows that while the third variation is much more likely to allow the first player a "perfect" win, it is still fairly unlikely unless the player is very accurate.

REFERENCES
1. D.J.H. Garling, *A Course In Galois Theory*, Cambridge University Press, Cambridge, 1986
2. B.V. Gnedenko, *The Theory of Probability*, Chelsea Publishing Company, New York, NY, 1966